BOARD OF STUDIES

NEW SOUTH WALES

2006

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Attempt Questions 1–7
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{dx}{49 + x^2}$. **2**

(b) Using the substitution $u = x^4 + 8$, or otherwise, find $\displaystyle\int x^3 \sqrt{x^4 + 8}\, dx$. **3**

(c) Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin 5x}{3x}$. **2**

(d) Using the sum of two cubes, simplify: **2**

$$\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} - 1,$$

for $0 < \theta < \dfrac{\pi}{2}$.

(e) For what values of b is the line $y = 12x + b$ tangent to $y = x^3$? **3**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = \sin^{-1}(x+5)$.

(i) State the domain and range of the function $f(x)$. **2**

(ii) Find the gradient of the graph of $y = f(x)$ at the point where $x = -5$. **2**

(iii) Sketch the graph of $y = f(x)$. **2**

(b) (i) By applying the binomial theorem to $(1+x)^n$ and differentiating, show that **1**

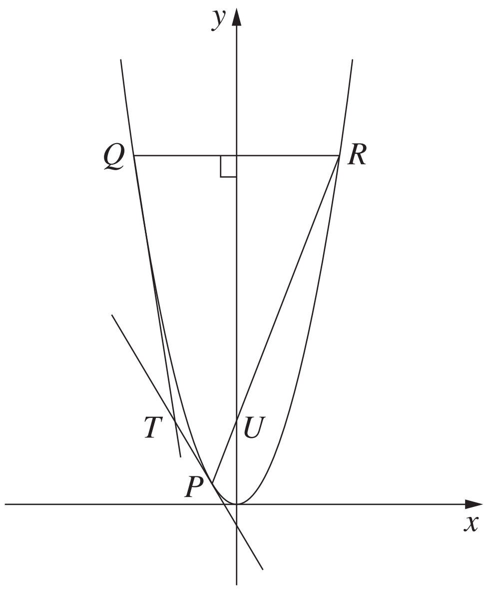
$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + r\binom{n}{r}x^{r-1} + \cdots + n\binom{n}{n}x^{n-1}.$$

(ii) Hence deduce that **1**

$$n3^{n-1} = \binom{n}{1} + \cdots + r\binom{n}{r}2^{r-1} + \cdots + n\binom{n}{n}2^{n-1}.$$

Question 2 continues on page 5

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ lie on the parabola $x^2 = 4ay$. The chord QR is perpendicular to the axis of the parabola. The chord PR meets the axis of the parabola at U.

The equation of the chord PR is $y = \frac{1}{2}(p + r)x - apr$. (Do NOT prove this.)

The equation of the tangent at P is $y = px - ap^2$. (Do NOT prove this.)

(i) Find the coordinates of U. **1**

(ii) The tangents at P and Q meet at the point T. Show that the coordinates of T are $(a(p + q), apq)$. **2**

(iii) Show that TU is perpendicular to the axis of the parabola. **1**

End of Question 2

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$. **2**

(b) (i) By considering $f(x) = 3\log_e x - x$, show that the curve $y = 3\log_e x$ and the line $y = x$ meet at a point P whose x-coordinate is between 1.5 and 2. **1**

(ii) Use one application of Newton's method, starting at $x = 1.5$, to find an approximation to the x-coordinate of P. Give your answer correct to two decimal places. **2**

(c) Sophie has five coloured blocks: one red, one blue, one green, one yellow and one white. She stacks two, three, four or five blocks on top of one another to form a vertical tower.

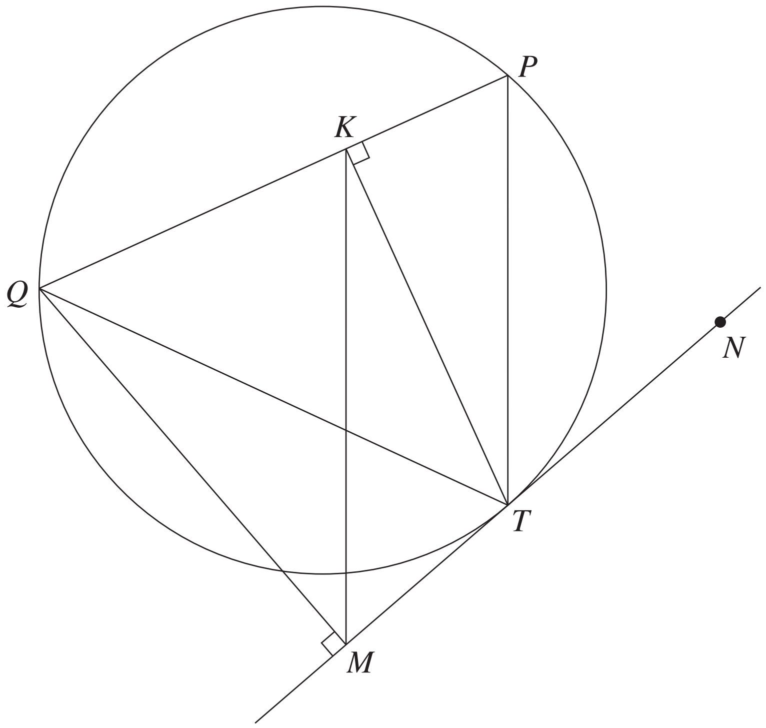
(i) How many different towers are there that she could form that are three blocks high? **1**

(ii) How many different towers can she form in total? **2**

Question 3 continues on page 7

Question 3 (continued)

(d)



The points P, Q and T lie on a circle. The line MN is tangent to the circle at T with M chosen so that QM is perpendicular to MN. The point K on PQ is chosen so that TK is perpendicular to PQ as shown in the diagram.

(i) Show that $QKTM$ is a cyclic quadrilateral. **1**

(ii) Show that $\angle KMT = \angle KQT$. **1**

(iii) Hence, or otherwise, show that MK is parallel to TP. **2**

End of Question 3

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Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) The cubic polynomial $P(x) = x^3 + rx^2 + sx + t$, where r, s and t are real numbers, has three real zeros, 1, α and $-\alpha$.

(i) Find the value of r. **1**

(ii) Find the value of $s + t$. **2**

(b) A particle is undergoing simple harmonic motion on the x-axis about the origin. It is initially at its extreme positive position. The amplitude of the motion is 18 and the particle returns to its initial position every 5 seconds.

(i) Write down an equation for the position of the particle at time t seconds. **1**

(ii) How long does the particle take to move from a rest position to the point halfway between that rest position and the equilibrium position? **2**

(c) A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$.

Initially $x = -2$ and the velocity, v, is -6.

(i) Show that $v^2 = 9x^2(1 + x)^2$. **2**

(ii) Hence, or otherwise, show that **2**

$$\int \frac{1}{x(1+x)}\,dx = -3t.$$

(iii) It can be shown that for some constant c, **2**

$$\log_e\left(1+\frac{1}{x}\right) = 3t + c. \qquad \text{(Do NOT prove this.)}$$

Using this equation and the initial conditions, find x as a function of t.

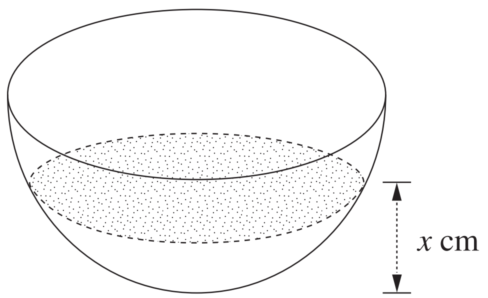
Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Show that $y = 10e^{-0.7t} + 3$ is a solution of $\dfrac{dy}{dt} = -0.7(y-3)$. **2**

(b) Let $f(x) = \log_e\left(1+e^x\right)$ for all x. Show that $f(x)$ has an inverse. **2**

(c)



A hemispherical bowl of radius r cm is initially empty. Water is poured into it at a constant rate of k cm^3 per minute. When the depth of water in the bowl is x cm, the volume, V cm^3, of water in the bowl is given by

$$V = \frac{\pi}{3}x^2(3r - x).$$ (Do NOT prove this.)

(i) Show that $\dfrac{dx}{dt} = \dfrac{k}{\pi x(2r-x)}$. **2**

(ii) Hence, or otherwise, show that it takes 3.5 times as long to fill the bowl to the point where $x = \dfrac{2}{3}r$ as it does to fill the bowl to the point where $x = \dfrac{1}{3}r$. **2**

Question 5 continues on page 11

Marks

Question 5 (continued)

(d) (i) Use the fact that $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ to show that **1**

$$1+\tan n\theta\ \tan(n+1)\theta=\cot\theta\left(\tan(n+1)\theta-\tan n\theta\right).$$

(ii) Use mathematical induction to prove that, for all integers $n\ge 1$, **3**

$$\tan\theta\ \tan 2\theta+\tan 2\theta\ \tan 3\theta+\cdots+\tan n\theta\ \tan(n+1)\theta=-(n+1)+\cot\theta\ \tan(n+1)\theta.$$

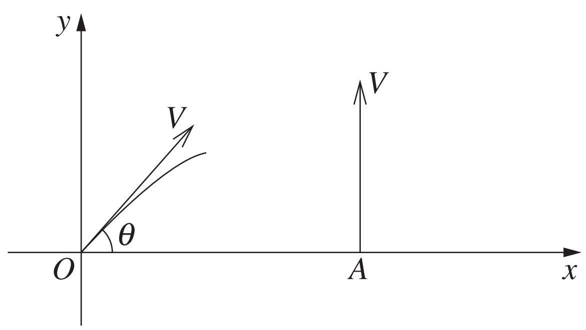
End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Two particles are fired simultaneously from the ground at time $t = 0$.

Particle 1 is projected from the origin at an angle θ, $0 < \theta < \frac{\pi}{2}$, with an initial velocity V.

Particle 2 is projected vertically upward from the point A, at a distance a to the right of the origin, also with an initial velocity of V.



It can be shown that while both particles are in flight, Particle 1 has equations of motion:

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

and Particle 2 has equations of motion:

$$x = a$$

$$y = Vt - \frac{1}{2}gt^2.$$

Do NOT prove these equations of motion.

Let L be the distance between the particles at time t.

Question 6 continues on page 13

Marks

Question 6 (continued)

(i) Show that, while both particles are in flight, **2**

$$L^2 = 2V^2t^2\left(1 - \sin\theta\right) - 2aVt\cos\theta + a^2.$$

(ii) An observer notices that the distance between the particles in flight first decreases, then increases. **3**

Show that the distance between the particles in flight is smallest when $t = \dfrac{a\cos\theta}{2V\left(1-\sin\theta\right)}$ and that this smallest distance is $a\sqrt{\dfrac{1-\sin\theta}{2}}$.

(iii) Show that the smallest distance between the two particles in flight occurs while Particle 1 is ascending if $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta\left(1-\sin\theta\right)}}$. **1**

(b) In an endurance event, the probability that a competitor will complete the course is p and the probability that a competitor will not complete the course is $q = 1 - p$. Teams consist of either two or four competitors. A team scores points if at least half its members complete the course.

(i) Show that the probability that a four-member team will have at least three of its members not complete the course is $4pq^3 + q^4$. **1**

(ii) Hence, or otherwise, find an expression in terms of q only for the probability that a four-member team will score points. **2**

(iii) Find an expression in terms of q only for the probability that a two-member team will score points. **1**

(iv) Hence, or otherwise, find the range of values of q for which a two-member team is more likely than a four-member team to score points. **2**

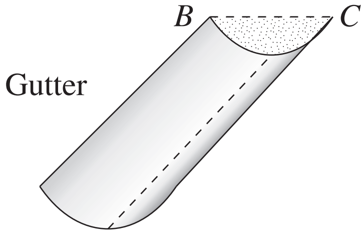
End of Question 6

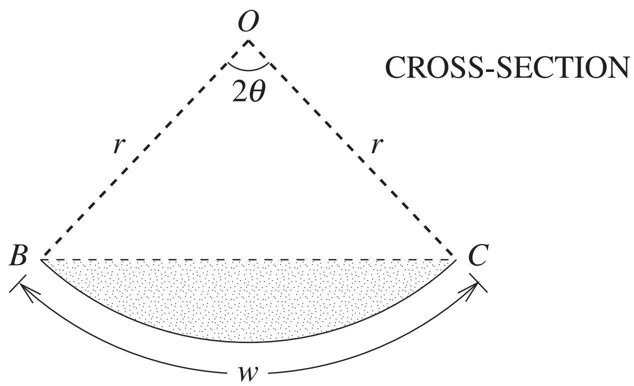
Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

A gutter is to be formed by bending a long rectangular metal strip of width w so that the cross-section is an arc of a circle.

Let r be the radius of the arc and 2θ the angle at the centre, O, so that the cross-sectional area, A, of the gutter is the area of the shaded region in the diagram on the right.





(a) Show that, when $0 < \theta \le \dfrac{\pi}{2}$, the cross-sectional area is **2**

$$A = r^2\left(\theta - \text{si } \ \theta\cos \ \right).$$

(b) The formula in part (a) for A is true for $0 < \theta < \pi$. (Do NOT prove this.) **3**

By first expressing r in terms of w and θ, and then differentiating, show that

$$\frac{dA}{d\theta} = \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$$

for $0 < \theta < \pi$.

Question 7 continues on page 15

Marks

Question 7 (continued)

(c) Let $g(\theta) = \sin\theta - \theta\cos\theta$. **3**

By considering $g'(\theta)$, show that $g(\theta) > 0$ for $0 < \theta < \pi$.

(d) Show that there is exactly one value of θ in the interval $0 < \theta < \pi$ for which $\frac{dA}{d\theta} = 0$. **2**

(e) Show that the value of θ for which $\frac{dA}{d\theta} = 0$ gives the maximum cross-sectional area. Find this area in terms of w. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$